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An elliptic equation with mixed skew boundary conditions

by

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Introduction

In reference [1], H.A. Lauwerier studies the behavior of solutions of the following boundary value problem for the half strip S : $0 \leq x \leq \pi$, $0 < y$:

$$
(0)\quad
\begin{cases}
\text{i) } F_{xx} + F_{yy} = q^2 F \text{ in } S \\
\text{ii) } F(x,y) \to 0 \text{ as } y \to \infty \\
\text{iii) } F(0,Y) = F(\pi,y) = 0 \\
\text{iv) } \sin\mu\pi\, F_x - \cos\mu\pi\, F_y = f(x) \text{ for } y=0.
\end{cases}
$$

Here q is real, and in most of the discussion $0 < \mu < 1/2$. Lauwerier gives a detailed analysis for $q=0$, showing how to obtain the solution as a sine series, and describing its behavior at the corners of S. He further relates the solution for $q \neq 0$ to that for $q=0$ by an integral equation $\psi + L\psi = F$ with a continuous bounded kernel $L(x,t)$, but does not investigate the solvability of the equation. In this note it is shown that for the values of q and μ considered, the integral equation always has a unique solution. We first indicate how problem (0) leads to the integral equation, then show the solvability of the integral equation, and finally indicate the consequences for the existence and uniqueness of solutions of problem (0). It is interesting to note that a solution is always unique, but that if $q^2 \leq -n^2$ there is no solution unless the f in (0,iv) is orthogonal to a certain set of n linearly independent functions; this is the result of Theorem 2. In case condition (0,ii) is relaxed, there is the related but less complete result of Theorem 3.

§ 1. Reduction to an integral equation

Our approach rests on an analysis of the eigenvalues of the homogeneous problem

$$
(1)\quad
\begin{aligned}
&\text{i)}\ F_{xx} + F_{yy} = \lambda F \quad \text{in } S \\
&\text{ii)}\ \lim_{y\to\infty} \int_0^\pi |F(x,y)|\,dx = 0 \\
&\text{iii)}\ F(0,y) = F(\pi,y) = 0 \ \text{for } y > 0 \\
&\text{iv)}\ \lim_{y\to 0} \int_0^\pi |\sin\mu\pi\, F_x(x,y) - \cos\mu\pi\, F_y(x,y)|\,dx = 0.
\end{aligned}
$$

F is required to be twice continuously differentiable in S.

This problem is amenable to separation, as indicated in reference [1]. For $y > 0$ F has a sine series $\sum_1^\infty B_n(y)\sin nx$, with $B_n(y) = \frac{2}{\pi}\int_0^\pi F(x,y)\sin nx\,dx$. Differentiating under the integral, applying (1,i), and integrating by parts yields $B_n'' = (\lambda+n^2)B_n$, while (1,ii) yields $B_n(\infty) = 0$. Thus $B_n = 0$ if $\lambda+n^2 \leq 0$, and $B_n(y) = \frac{1}{n} b_n \exp(-\sqrt{\lambda+n^2}\,y)$ otherwise; here the root is to be taken with positive real part. Thus the solution of (1,i)-(1,iii) can be written

$$(2)\qquad F(x,y) = \sum{}^{*}(b_n/n)\sin nx\ e^{-n\alpha_n y},$$

where $\sum^*$ is the sum over all $n \geq 1$ such that $\lambda+n^2$ is not ≤ 0, and α_n is defined by

$$(3)\qquad \alpha_n^2 = 1 + \lambda n^{-2}, \quad \mathrm{Real}(\alpha_n) > 0.$$

In order to relate the condition (1,iv) to the integral equation in question, we introduce the following expansions and operators:

$$(4)\qquad \varphi(x) = \sum_1^\infty b_n \sin(nx+\mu\pi)$$

$$(5)\qquad A\varphi(x) = \sum{}^{*} b_n\alpha_n \sin(nx+\mu\pi)$$

$$(6)\qquad L\varphi(x) = \sum{}^{*}(\alpha_n^{-1}-1)b_n \sin\mu\pi\cos nx.$$

Then formally

$$(7)\qquad \sin\mu\pi\, F_x(x,0) - \cos\mu\pi\, F_y(x,0) = A\varphi + LA\varphi.$$

According to reference [1], the coefficients in the series (4) can be calculated by

(8) $$b_n = \frac{1}{\pi}\int_0^{\pi}\varphi(t)k_n(t)dt,$$

where $\{k_n\}$ is an appropriate system of functions biorthogonal with respect to $\{\sin(nx+\mu\pi)\}$. Further, $|k_n(t)| \leq 2+4\sin\mu\pi$; and if φ satisfies a Hölder condition $|\varphi(x)-\varphi(x')| \leq c|x-x'|^{\gamma}$ for some c, $0<\gamma<1$, it is shown that

(9) $$b_n = -a_o\, e_n(-2\mu) + a_n \qquad \text{(formula 2.23 in [1])}$$

wherein

(10) $$\begin{cases} a_o = \frac{1}{\pi}\int_0^{\pi}(\operatorname{tg}\frac{t}{2})^{1-2\mu}\varphi(t)dt \\ a_n = O(n^{-1-2\mu}) \\ e_n = O(n^{2\mu-1}), \end{cases}$$

and in fact $e_n(\alpha)$ is defined by

(11) $$(1+z)^{\alpha}\,(1-z)^{-\alpha} = \sum_0^{\infty} e_n(\alpha)z^n.$$

From formulas (6) and (8) it follows that L is an integral operator $L\varphi(x) = \frac{1}{\pi}\int_0^{\pi}L(x,t)\,\varphi(t)dt$, with $L(x,t) = \sin\mu\pi\sum^{*}(\alpha_n^{-1}-1)\cos nx\; k_n(t)$. To see that this agrees with the kernel L(x,t) given in reference [1], formula 5.13, use formula 1.16 of [1] with $\lambda=q^2$.

We observe that, since $|\cos nx - \cos nx'| \leq 2^{1-\gamma}n^{\gamma}|x-x'|^{\gamma}$ for $0\leq\gamma\leq 1$, and $\alpha_n^{-1}-1 = O(n^{-2})$, the kernel L(x,t) satisfies a Hölder condition for $\gamma<1$, $|L(x,t)-L(x',t)| \leq c|x-x'|^{\gamma}$, with c independent of t. Thus if φ is integrable, then $L\varphi$ satisfies a Hölder condition. We can show further that φ satisfies a Hölder condition if and only if $A\varphi$ does. For according to (5) $A\varphi = \varphi + K\varphi$, where K is the operator with kernel $K(x,t) = \sum^*(\alpha_n-1)\sin(nx+\mu\pi)k_n(t) - \sum_{n^2\leq -\lambda} \sin(nx+\mu\pi)k_n(t)$, which can be shown to satisfy a Hölder condition in the same way as $L(x,t)$; since $A\varphi - \varphi$ is always Hölder continuous, φ is Hölder continuous if and only if $A\varphi$ is.

§ 2. Assumption of boundary values

In order to investigate $\varphi + L\varphi = f$ in connection with (1), we need the following result.

Theorem 1. *Let $\varphi(x)$ be Hölder continuous, the b_n be defined by (8), and F be defined by (2). Then, as $y \to 0$, we have $F(x,y) \to F(x,0)$ uniformly in $0 \leq x \leq \pi$, and $F_x(x,y)$ and $F_y(x,y)$ converge in $L^p(0,\pi)$ to $\sum^* b_n \cos nx$ and $-\sum^* b_n\alpha_n \sin nx$ respectively, for $p < 1/2\mu$.*

Proof. Since (9) and (10) yield $\sum^* |b_n|/n < \infty$, the uniformity of $F(x,y) \to F(x,0)$ is clear. As for $F_y(x,y)$ we can write it as

$$
\begin{aligned}
F_y(x,y) &= a_o \sum_1^\infty e_n(-2\mu)\sin nx\, e^{-ny} - a_o \sum_{n^2\leq -\lambda} e_n(-2\mu)\sin nx\, e^{-ny} \\
&\quad + a_o \sum^* e_n(-2\mu)\sin nx(\alpha_n e^{-n\alpha_n y} - e^{-ny}) \\
&\quad - \sum^* \alpha_n a_n \sin nx\, e^{-n\alpha_n y} \\
&= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 .
\end{aligned}
$$

Since $a_n = O(n^{-1-2\mu})$, Σ_4 converges uniformly to its boundary value; and since for $0 \leq y \leq 1$, $e_n(-2\mu)(\alpha_n e^{-n\alpha_n y} - e^{-ny}) = O(n^{2\mu-2})$, Σ_3 also converges uniformly; while Σ_2 is a finite sum. To treat Σ_1 we write r for e^{-y}, and find by taking the imaginary part of formula (11) that Σ_1 is the Abel mean $f(r,x)$ of the odd function $f(1,x)$ defined for $0 \leq x \leq \pi$ by $f(1,x) = -a_o \sin\mu\pi(\operatorname{tg}\frac{x}{2})^{2\mu}$. It follows from a familiar result in Fourier series (reference [2], p.87) that as $y \to 0$, $\sum_1$ converges in L^p to $a_o \sum_1^\infty e_n(-2\mu)\sin nx$ for $p < 1/2\mu$, since $f(1,x)$ is in L^p for these values of p. A similar treatment yields the same result for F_x.

§ 3. Solvability of $\varphi + L\varphi = f$

Establishing the solvability of $\varphi + L\varphi = f$ for arbitrary f is equivalent to showing that $\varphi + L\varphi = 0$ has no solution but $\varphi=0$, because of the familiar Fredholm alternative. Suppose then that $\varphi + L\varphi = 0$ for some φ in L^1. As we have seen, φ satisfies a Hölder condition, so that it has the expansion (4) with coefficients given by (8), and yields the solution (2) of problem (1). Condition (1,iv) follows formally from (7), and strictly from Theorem 1.

Form now, for $y > 0$, the function

$$I(y) = \int_0^\pi (F F_y^* + F_y F^*)dx$$

$$= -\pi \sum{}^* |b_n|^2 n^{-1} \operatorname{Real}(\alpha_n)\exp(-2y \operatorname{Real}(n\alpha_n)),$$

where F^* is the complex conjugate of F. Since for the values of n occurring in $\sum^*$, Real $(\alpha_n) > 0$, we can complete the proof by showing $I(y) \to 0$ as $y \to 0$, as follows. From $I(y) \to 0$ we conclude that $b_n=0$ if $n^2 + \lambda$ is not ≤ 0, so $L\varphi = \sum^*(\alpha_n^{-1} - 1)b_n \sin\mu\pi \cos nx = 0$. Since $\varphi + L\varphi = 0$, we have $\varphi = 0$.

To show $I(y) \to 0$, note first that for $y > 0$ an integration by parts yields

$$\int_0^\pi (F F_x^* + F^* F_x)dx = 0.$$

Thus
$$\cos\mu\pi \, I(y) = \int_0^\pi (\cos\mu\pi \, F_y^* - \sin\mu\pi \, F_x^*) F \, dx$$
$$+ \int_0^\pi (\cos\mu\pi \, F_y - \sin\mu\pi \, F_x) F^* dx.$$

Since F is bounded and the terms in parentheses converge to zero in L^1 (by Theorem 1), it follows that $\lim_{y \to 0} I(y) = 0$. This completes the proof that $\varphi + L\varphi = 0$ has no non-zero solution.

Since $\varphi + L\varphi = f$ has a unique solution for every f, there is a uniquely defined operator $(I+L)^{-1}$.

§ 4. Application to the inhomogeneous problem

We turn now to the original problem (0), with $q^2 = \lambda$. Conditions (0,ii) and (0,iv) are to be interpreted respectively as $\lim_{y\to\infty} \int_0^\pi |F(x,y)|\,dx = 0$ and $\lim_{y\to 0} \int_0^\pi |\sin\mu\pi\, F_x(x,y) - \cos\mu\pi\, F_y(x,y) - f(x)|\,dx = 0$. The sequence $\{k_n(x)\}$ is biorthogonal to $\{\sin(nx+\mu\pi)\}$.

Theorem 2. (i) Any solution of (0) is unique;

ii) $(I+L)^{-1}[\sin\mu\pi\, F_x(x,y) - \cos\mu\pi\, F_y(x,y)]$ is orthogonal to $k_n(x)$ for each $n^2 \leq -\lambda$;

iii) if f is Hölder continuous and $(I+L)^{-1}f$ is orthogonal to k_n for $n^2 \leq -\lambda$, then there is a solution of (0).

Proof.

ii) Let $F(x,y) = \sum^* b_n n^{-1} \sin nx\, e^{-n\alpha_n y}$ be an arbitrary solution of (0,i)-(0,iii). The formula

$$(12)\quad (I+L)^{-1}[\sin\mu\pi\, F_x - \cos\mu\pi\, F_y] = \sum^* \alpha_n b_n \sin(nx+\mu\pi) e^{-n\alpha_n y}$$

can be checked by applying I+L to both sides.
It follows immediately that $(I+L)^{-1}[\sin\mu\pi\, F_x - \cos\mu\pi\, F_y]$ is orthogonal to $k_n(x)$ if n does not occur in $\sum^*$, i.e. if $n^2 \leq -\lambda$.

i) From formula (12) we get for n^2 not $\leq -\lambda$ that b_n is uniquely determined by $b_n = (\pi\alpha_n)^{-1} \lim_{y\to 0} \int_0^\pi k_n(x)(I+L)^{-1} [\sin\mu\pi\, F_x - \cos\mu\pi\, F_y]\,dx$; hence F is unique.

ii) If $(I+L)^{-1} f$ is orthogonal to the appropriate k_n, then $(I+L)^{-1} f = \sum^* c_n \sin(nx+\mu\pi)$. Setting $b_n = c_n/\alpha_n$ and defining F by formula (2), we obtain the solution of (0).

Thus a unique solution exists for every f unless $\lambda \not\leq -1$; while if $\lambda \leq -N^2$ the solution is still unique, but exists if and only if f is orthogonal to a certain set of N functions, namely $Kk_1,\dots,Kk_N$, where K is the adjoint of $(I+L)^{-1}$.

§ 5. A weaker condition at ∞

It is interesting to see what the analog of Theorem 2 becomes if the condition at ∞ is weakened, i.e. if we consider the problem

$$\text{(13)}\quad \begin{aligned} &\text{i)}\quad F_{xx} + F_{yy} = \lambda F \text{ in } S \\ &\text{ii)}\quad \int_0^{\pi} |F(x,y)|\,dx \text{ bounded for } 0 \leq y < \infty \\ &\text{iii)}\quad F(0,y) = F(\pi,y) = 0 \text{ for } y > 0 \\ &\text{iv)}\quad \lim_{y \to 0} \int_0^{\pi} |\sin \mu\pi\, F_x(x,y) - \cos \mu\pi\, F_y(x,y) - g|\,dx = 0. \end{aligned}$$

For this our results are not as explicit as for (0), but clearly indicate the relative strengthening of existence and weakening of uniqueness that is to be expected. We consider only $\lambda \leq -1$, since otherwise (13) and (0) are equivalent.

Theorem 3. Let $\lambda \leq -1$, and N be the greatest integer such that $N^2 < -\lambda$ Let N_1 = the number of solutions of the homogeneous problem (13) (g=0), and N_2 the number of conditions to be fulfilled by g in order for a solution to exist. Then $N_1 - N_2 = N$.

Proof. We consider the case that $-\lambda = (N+1)^2$; if $-\lambda$ is not a square the proof is quite similar but somewhat simpler. The method is to reduce problem (13) to problem (0) by subtracting the new solutions, which are bounded but do not vanish at ∞.

For $n > N+1$ let $\alpha_n = (1+\lambda n^{-2})^{1/2}$, and for $n \leq N$ let $\beta_n = (-1-\lambda n^{-2})^{1/2}$. Then the general solution of (13,i) - (13,iii) is $F(x,y) = \sum_1^N n^{-1} \sin nx(c_n \cos n\beta_n y + d_n \sin n\beta_n y) + (N+1)^{-1} c_{N+1} \sin(N+1)x + \sum^* b_n n^{-1} \sin nx\, e^{-n\alpha_n y}$, where now $\sum^* = \sum_{N+2}^{\infty}$. Let $F = F^{(1)} + F^{(2)}$ with $F^{(2)} = \sum^* b_n n^{-1} \sin nx\, e^{-n\alpha_n y}$, and let $f_1(x) = \sin\mu\pi\, F_x^{(1)}(x,0) - \cos\mu\pi\, F_y^{(1)}(x,0)$. Then $F^{(2)}$ is a solution of problem (0) with $f = g - f_1$. According to Theorem 2 this is possible if and only if $(I+L)^{-1}(g-f_1)$ is orthogonal to k_n for $n \leq N+1$, i.e. if and only if $\int_0^\pi k_n(I+L)^{-1} f_1 = \int_0^\pi k_n(I+L)^{-1} g$ for $1 \leq n \leq N+1$. Since $f_1 = \sum_1^{N+1} c_n \cos nx + \sum_1^N \beta_n d_n \sin nx$, f_1 ranges over a $2N+1$ dimensional space, and the problem (13) is equivalent to solving $N+1$ equations in $2N+1$ unknowns, from which the conclusion follows.

References

[1] H.A. Lauwerier, "On Certain Trigonometrical Expansions", Journal of Mathematics and Mechanics, vol.8 (1959), pp.419-432.

[2] A. Zygmund, Trigonometrical Series, Warsaw, 1935.